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A direct method for solving fractional order variational problems by hat basis functions

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ABSTRACT

This paper presents a numerical technique for solving a class of fractional variational problems using a direct method based on operational matrix of generalized hat basis function. The fractional derivative is defined in the Caputo sense.

Minimization of such functional leads to a set of algebraic equations which are solved using an appropriate numerical technique. For the class of problems considered, the numerical solution can be obtained directly from the functional and there is no need to solve the fractional Euler-Lagrange equations.

Numerical examples are introduced combined with their approximate solutions and comparison with other numerical approach for conforming the accuracy and applicability of the proposed method.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivative and integrals of arbitrary orders (including complex orders), and their applications in science, engineering, mathematics, economies, and other fields [1].

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields [2].

Most fractional order differential equations do not have exact solutions, so approximate and numerical techniques must be used. Several numerical and approximate methods to solve fractional order differential equations have been given such as Adomian decomposition method [3], variational iteration method [4], homotopy analysis method [5], homotopy perturbation method [6], collocation method [7,8], wavelet method [9], finite element method [10] and spectral tau method [11].

Recently, the operational matrices of fractional derivatives and integrals have been derived for some types of orthogonal polynomials such as, the Legendre polynomials [12–16], Chebyshev polynomials [17,18], Jacobi polynomials [19–23] and Laguerre polynomials [24] and used to solve several types of fractional order differential equations see [25,26].

A fractional calculus of variations problem is a subtopic of fractional calculus and it is a problem in which either the objective functional or the constraint equation or both contain at least one fractional derivative term [27].

This occurs naturally in many problems of physics, mechanics and engineering in order to provide more accurate models of physics phenomena (see [28,29]).

It is remarkable to refer that the starting point of fractional calculus of variation appear to be the Refs. [30,31], where Riewe developed the non concentrative Lagrangian, Hamiltonian, and other concepts of classical mechanics using fractional derivative.

After that numerous works have been dedicated to the fractional calculus of variation see for example [32–37].

In recent years, many analytical and numerical techniques have been used for obtaining the solution of fractional variational problems among them for instance [27,38–44], (see also [45–48]).

Babolian and Mordad [49] have used hat basis function for solving system of linear and nonlinear integral equations of the second kind by the hat functions. In [50], Tripathi et al. have used generalized hat basis functions to obtain approximate solutions of linear and nonlinear fractional differential equations. Heydri et al. [51] have used generalized hat basis functions for solving stochastic it $\hat{o}$ -volterra integral equations. In [52] Heydri et al. used generalized hat basis functions for solving nonlinear stochastic it $\hat{o}$ integral equations. In [53] Heydri et al. have used hat basis functions for solving a class of fractional optimal control problems.

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In this paper a generalized hat basis function will be used for solving a class of fractional variational problems.

This paper is organized as follows: In Section 2 some definitions of fractional order derivatives and integration are investigated, in Section 3 we describe the generalized hat basis functions and their properties. In Section 4 operational matrices of fractional and integer order integration for the generalized hat functions is given. In Section 5 the proposed method is described for solving fractional order variational problems by generalized hat basis functions. In Section 6 we introduce some illustrative examples to show the accuracy and efficiency of the proposed method. Finally a conclusion is drawn in Section 7.

2. Fractional order integration and derivative

There are various definitions of fractional integration and derivative such as Riemann-Liouville and Caputo definitions. Comparatively, the Caputo derivative has certain advantages when trying to model real-world phenomena with fractional differential equations.

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as follows:

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad x > 0, \quad \alpha \in \mathbb{R}^+$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:

$${}_0^c D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau & m-1 < \alpha < m \\ \frac{d^m}{dx^m} f(x) & \alpha = m \end{cases}$$

for $\alpha > 0$, we have the following properties of the Caputo fractional derivative:

(i) ${}_0^c D_x^{\alpha}(I_x^{\alpha} f(x)) = f(x)$..
(ii) $I_x^{\alpha}({}_0^c D_x^{\alpha} f(x)) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}$.
(iii) ${}_0^c D_x^{\alpha}(C) = 0, C \in \mathbb{R}$.
(iv) ${}_0^c D_x^{\alpha}(x^{\gamma}) = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha} & \gamma \in \{0,1,2,3,\ldots\}, \quad \gamma \geqslant [\alpha] \\ 0 & \gamma \in \{0,1,2,3,\ldots\}, \quad \gamma < [\alpha] \end{cases}$

where $[\alpha]$ is the floor function of α.

3. Generalized hat functions and their properties

Usually hat functions are defined on the interval [0,1]. These are continuous functions with shape of hats, when plotted on two dimensional planes. In this section the generalized hat functions [50] are considered on the interval [0,T]. The interval [0,T] is divided into n subintervals $[ih, (i+1)h], i = 0,1,2,\ldots n-1$; of equal lengths h where $h = \frac{T}{n}$. The generalized hat functions family of first $(n+1)$ hat functions is defined as follows:

$$\psi_0(x) = \begin{cases} \frac{h-x}{h}, & 0 \leqslant x < h \\ 0, & otherwise \end{cases} \tag{1}$$

$$\psi_i(x) = \begin{cases} \frac{x-(i-1)h}{h}, & (i-1)h \leqslant x \leqslant ih \\ \frac{(i+1)h-x}{h}, ih \leqslant x \leqslant (i+1)h, & i = 1,2,\ldots,n-1 \\ 0, & otherwise \end{cases} \tag{2}$$

$$\psi_n(x) = \begin{cases} \frac{x-(T-h)}{h}, & T-h \leqslant x \leqslant T \\ 0, & otherwise \end{cases} \tag{3}$$

These $\psi_i(x)^s \in L^2[0,T]$ and are linearly independent. Using the definition of generalized hat functions, we have the following observations.

$$\psi_i(kh) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \tag{4}$$

and

$$\psi_i(x)\psi_j(x) = 0, \quad |i-j| \geqslant 2 \tag{5}$$

Any arbitrary function $f \in L^2[0,T]$ is approximated in vector form as:

$$f(x) \cong \sum_{i=0}^{i=n} f_i \psi_i(x) = F_{n+1}^T \psi_{n+1}(x) = \psi_{n+1}^T F_{n+1} \tag{6}$$

where

$$F_{n+1} = [f_0, f_1, f_2, \ldots, f_n]^T \tag{7}$$

and

$$\Psi_{n+1}(x) = [\psi_0(x), \psi_1(x), \psi_2(x), \ldots \psi_n(x)]^T \tag{8}$$

The importance aspect of using the generalized hat function in the approximation of function $f(x)$, lies in the fact that the coefficient f_i in Eq. (6), are given by:

$$f_i = f(ih), i = 0,1,2,\ldots,n \tag{9}$$

Moreover if $T = 1$ then from Eq.(5), we have

$$\int_0^1 \Psi_{n+1}(x)\Psi_{n+1}^T(x)dt = K_{n+1Xn+1}$$

where $K_{n+1Xn+1} = \frac{h}{6}\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix}$

4. Fractional and integer order operational matrices of the integration for generalized hat functions

This section discussed the derivation [50] of operational matrices of fractional and integer order of the integration for generalized hat functions in the subsections 4.1 and 4.2 respectively.

4.1. Operational matrix of integer order integration of the generalized hat functions

Since $\int_0^x \psi_i(x)dx \in L^2[0,T]$, Eq. (6) is used to approximate it in the terms of the generalized hat basis functions as:

$$\int_0^x \psi_i(x)dx \cong \sum_{j=0}^{n} a_{ij}\psi_j(x)dx, \quad i = 0,1,2,\ldots,n \tag{10}$$

The coefficients a_{ij} and according to Eq. (9) are given by

$$a_{ij} = \int_0^{jh} \psi_i(x)dx, \quad j = 0,1,2,\ldots,n.$$

From Eqs. (8) and (10) we have

$$\int_0^x \Psi_{n+1}dx \cong P_{n+1}\Psi_{n+1}(x)$$

The coefficients a_{ij} will form a $(n+1)\times(n+1)$ matrix P_{n+1} which is given by

$$P_{n+1}=\left(\frac{h}{2}\right)\begin{bmatrix}0&1&1&1&\cdots&1&1&1\\0&1&2&2&\cdots&2&2&2\\0&0&1&2&\cdots&2&2&2\\0&0&0&1&\cdots&2&2&2\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&\cdots&1&2&2\\0&0&0&0&\cdots&0&1&2\\0&0&0&0&&0&0&1\end{bmatrix}$$

4.2. Operational matrix of fractional order integration of the generalized hat functions

The fractional integration of order α of the vector $\psi_{n+1}(x)$ defined in (8) can be expressed as [50]:

$$(I^{\alpha}\Psi_{n+1})(x)=P^{\alpha}_{n+1}\Psi_{n+1}(x)$$

where the $(n+1)\times(n+1)$ matrix P^{α}_{n+1} is called the operational matrix of fractional integration [50,53]:

where

$$P^{\alpha}_{n+1}=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\begin{bmatrix}0&\theta_1&\theta_2&\theta_3&\cdots&\theta_n\\0&1&\vartheta_1&\vartheta_2&\cdots&\vartheta_{n-1}\\0&0&1&\vartheta_1&\cdots&\vartheta_{n-2}\\0&0&0&1&\cdots&\vartheta_{n-3}\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&0&1\end{bmatrix}$$

where $\theta_k=k^{\alpha}(\alpha-k+1)+(k-1)^{\alpha+1},k=1,2,3,\ldots,n;$ and $\vartheta_k=(k+1)^{\alpha+1}-2k^{\alpha+1}+(k-1)^{\alpha+1},k=1,2,3,\ldots,n-1.$

5. The direct approach for solving fractional order variational problems using operational matrix of generalized hat basis function

In this section we shall consider the problem of exterimization of a functional J of the form:

$$J[y(x)]=\int_0^1 F[x,y(x),{}^c_0D^{\alpha}_x y(x)]dx,\quad 0<\alpha\leqslant 1 \tag{11}$$

Satisfying the condition $y(0)=y_0$ and $y(1)$ is considered to be undetermined. ${}^c_0D^{\alpha}_x y$ refers to the fractional order derivative in the Caputo sense. The regular method for solving problem (11) is through the Euler-Lagrange equation, [54]:

$$\frac{\partial F}{\partial y}+{}^c_xD^{\alpha}_1\frac{\partial F}{\partial {}^c_0D^{\alpha}_x y}=0$$

And

$$\left(\frac{\partial F}{\partial {}^c_0D^{\propto}_1 y}\right)\Big|_{x=1}=0 \tag{12}$$

This section mainly uses the operational matrix of generalized hat functions to establish ψ_{n+1}. Now we approximate ${}^c_0D^{\alpha}_x y$ in terms of hat basis functions as follows:

$${}^c_0D^{\alpha}_x y\approx C^T_{n+1}\Psi_{n+1} \tag{13}$$

where $C_{n+1}=[c_0,c_1,c_2,\ldots,c_n]^T$ and $\Psi_{n+1}(x)=[\psi_0(x),\psi_1(x),\psi_2(x),\ldots,\psi_n(x)]^T$. Applying I^{α}_x to the both sides of Eq. (13), hence $y(x)$ can be expressed as:

$$y(x)\approx C^T_{n+1}P^{\alpha}_{n+1}\Psi_{n+1}(x)+y(0)$$

i.e.,

$$y(x)\approx C^T_{n+1}P^{\alpha}_{n+1}\Psi_{n+1}(x)+y_0 \tag{14}$$

The other terms in the functional of Eq. (11) are known functions of the independent variable x and can be expanded into hat basis functions through substitution, and finally we have:

$$J=J(c_0,c_1,c_2,\ldots,c_n) \tag{15}$$

The original extremization of a fractional variational problem shown in Eq. (11) becomes the extremization of a functional of a finite set of variables in Eq. (15). Taking the partial derivative of J with respect to c_i and setting them equal to zero, we obtain

$$\frac{\partial J}{\partial c_i}=0.\quad i=0,1,2.,\ldots,n$$

Solving for c_i, and substituting into Eq. (14), one can reach to the desired solution of Eq. (11).

6. Numerical examples

In order to have a comparison, a direct Haar wavelet method of Eq. (11) is briefly derived based on the previous work [44]. Assuming

$${}^c_0D^{\alpha}_x y(x)=\sum_{i=0}^{\infty}e_i h_i(x)$$

where $h_i(x)$ indicates the Haar basis function [55]. Taking finite terms as an approximation, we have

$${}^c_0D^{\alpha}_x y(x)=\sum_{i=0}^{m-1}e_i h_i(x)=E^T_m H_m(x)$$

Where $E^T_m=[e_0,e_1,e_2,\ldots,e_{m-1}]^T$ and $H_m(x)$

$$=[h_0(x),h_1(x),h_2(x),\ldots,h_{m-1}(x)]^T \tag{16}$$

Applying I^{α}_x to the both sides of Eq. (16), thus $y(x)$ can be written as

$$y(x)=E^T_m G^{\alpha}_{m\times m}H_m(x)+y_0 \tag{17}$$

where $G^{\alpha}_{m\times m}$ is called the Haar wavelet operational matrix of fractional order integration [56]

Substituting (16) and (17) in the functional (11) and therefore we have

$$J=J(e_0,e_1,e_2,\ldots,e_n)$$

Set $\frac{\partial J}{\partial e_i}=0$, $i=0,1,\ldots,m-1$.

And solving for e_i and then substituting into (17) in order to get the desired solution.

Example 1. Consider the functional:

$$J[y(x)]=\int_0^1\left[\frac{1}{2}\left({}^c_0D^{\alpha}_x y(x)\right)^2-y(x)\right]dx.\quad 0<\alpha\leqslant 1 \tag{18}$$

and the boundary condition

$$y(0)=y_0 \text{ and } y\ (1) \text{ is unspecified} \tag{19}$$

First, we approximate ${}^c_0D^{\alpha}_x y(x)$ in terms of hat basis functions as follows:

$${}^c_0D^{\alpha}_x y=C^T_{n+1}\Psi_{n+1}(x) \tag{20}$$

Here, we shall consider $y_0=0$ and n = 8 and therefore h = 0.125.

Now, upon taking I^{α}_x to the both sides of Eq. (20) thus we get:

$$y(x)=C^T_{n+1}P^{\alpha}_{n+1}\Psi_{n+1}(x) \tag{21}$$

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The other condition that we have is given by Eq. (12) and according to our example, we get:

$$ {}_0^cD_x^\alpha y(x)\big|_{x=1} = 0 \tag{22}$$

which implies that:

$$C_9^T\Psi_9(1) = 0$$

Therefore

$$C_8 = 0 \tag{23}$$

Substituting Eqs. (20), (21) and (23) into Eq. (18) we get:

$$J[y(x)] = \int_0^1 \left[\frac{1}{2}\left(C_9^T\Psi_9(x)\Psi_9^T(x)C_9\right) - C_9^T P_9^\alpha \Psi_9(x)\right]dx$$

Hence:

$$J[y(x)] = \frac{1}{2}C_9^T\int_0^1 \Psi_9(x)\Psi_9^T(x)dxC_9 - C_9^T P_9^\alpha\int_0^1 \Psi_9(x)dx \tag{24}$$

Or

$$J[y(x)] = \frac{1}{2}C_9^T K_{9X9}C_9 - C_9^T P_9^\alpha \begin{bmatrix} \frac{0.125}{2} \\ 0.125 \\ 0.125 \\ 0.125 \\ 0.125 \\ 0.125 \\ 0.125 \\ 0.125 \\ \frac{0.125}{2} \end{bmatrix}$$

Following Fig. 1 represent the approximate solution of Eq. (18) for different values of α with a comparison with the exact solution when $\alpha = 1$ which was given in [27] as:

$$y(x) = x(1 - \frac{x}{2})$$

Following Table 1 represent a comparison between the numerical solution of Eq. (18) using the proposed method and the Haar wavelet method with the exact solution when $\alpha = 1$.

Example 2. Find the extremal of the following functional:

$$J[y(x)] = \int_0^1 [({}_0^cD_x^\alpha y(x))^2 + x\,{}_0^cD_x^\alpha y(x)]dx. \quad 0 < \alpha \leqslant 1 \tag{25}$$

And the boundary condition

$y(0) = y_0$ and y(1) is unspecified

Table 1
Comparison between the numerical solution of Eq. (18) using the proposed method and Haar wavelet method with the exact solution when $\alpha = 1$.

x	The proposed method n = 8	Haar wavelet method m = 8	Exact solution
0	0	0.058	0
0.125	0.116	0.168	0.117
0.25	0.218	0.317	0.219
0.375	0.304	0.34	0.305
0.5	0.374	0.535	0.375
0.625	0.429	0.45	0.430
0.75	0.468	0.561	0.469
0.875	0.491	0.623	0.492

We also approximate ${}_0^cD_x^\alpha y(x)$ in terms of hat basis functions with n = 8, $y_0 = 0$ and hence h = 0.125 as follows:

$$ {}_0^cD_x^\alpha y = C_{n+1}^T\psi_{n+1}(x) \tag{26}$$

Hence:

$$y(x) = C_{n+1}^T P_{n+1}^\alpha \psi_{n+1}(x) \tag{27}$$

There is a variable x involved in Eq. (25) explicitly and it can be expanded in terms of hat basis function as:

$$x = d_{n+1}^T\psi_{n+1}(x) \tag{28}$$

where

$$d_{n+1} = [0 \quad 0.125 \quad 0.25 \quad 0.375 \quad 0.5 \quad 0.625 \quad 0.75 \quad 0.875 \quad 1]^T$$

By the transversality condition we have:

$$2\,{}_0^cD_x^\alpha y(x) + x\big|_{x=1} = 0$$

Which implies that:

$$2C_{n+1}^T\Psi_{n+1}(x) + x\Big|_{x=1} = 0$$

$$C_{n+1}^T\Psi_{n+1}(1) = -\frac{1}{2}$$

Hence:

$$C_8 = -\frac{1}{2} \tag{29}$$

Substituting Eqs. (26), (28) and (29) into Eqs. (25) yields:

$$J[y(x)] \approx \int_0^1 \left[(C_9^T\Psi_9(x)\Psi_9^T(x)C_9) + C_9^T\Psi_9(x)\Psi_9^T(x)d_9\right]dx$$

Or

$$J[y(x)] = C_9^T K_{9X9}C_9 + C_9^T K_{9X9}d_9$$

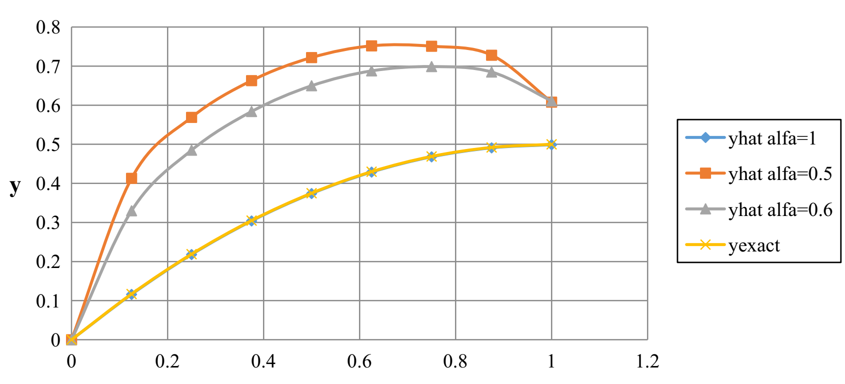

Figure 1. Represent the numerical solution of Example (1) for different values of α with the exact solution.

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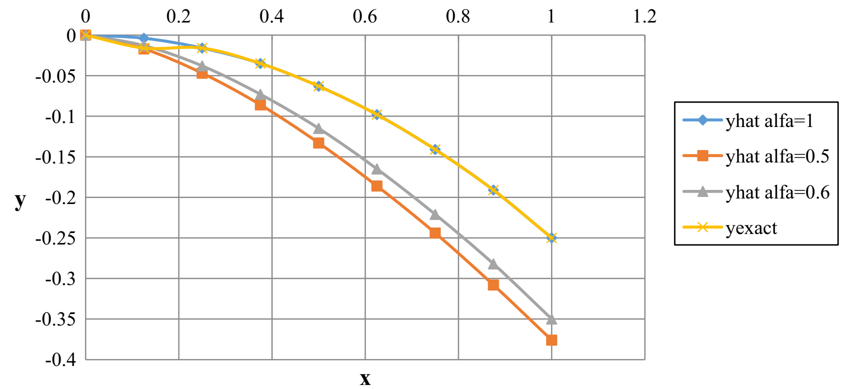

Figure 2. Represent the numerical solution of Example (2) for different values of α with the exact solution.

Table 2
Comparison between the numerical solution of Eq. (25) using the proposed method and Haar wavelet method with the exact solution when $\alpha = 1$.

x	The proposed method n = 8	Haar wavelet method m = 8	Exact solution
0	0	0.001062	0
0.125	−0.003909	−0.008938	−0.003906
0.25	−0.016	−0.029	−0.016
0.375	−0.035	−0.049	−0.035
0.5	−0.063	−0.108	−0.063
0.625	−0.098	−0.120	−0.098
0.75	−0.141	−0.186	−0.141
0.875	−0.191	−0.222	−0.191

Following Fig. 2 represent the approximate solution of Eq. (25) for different values of α with a comparison with the exact solution when $\alpha = 1$ which was given in [57] as:

$$y(x) = -\frac{x^2}{4}$$

Following Table 2 represent a comparison between the numerical solution of Eq. (25) using the proposed method and the Haar wavelet method with the exact solution when $\alpha = 1$.

7. Conclusions

In this study a direct method based on operational matrix of generalized hat basis function was applied to solve fractional order variational problems. The fractional derivative is defined in the Caputo sense. The underlying problem was reduced under the proposed method to the problem of solving a set of algebraic equations which are solved by using an appropriate numerical technique.

The numerical solution for the class of problems considered can be obtained directly from the functional and there is no need to solve the fractional Euler-Lagrange equations.

By comparing the numerical solution using the proposed method with the exact solution and that of Haar wavelet method, we demonstrate the accuracy and efficiency of the proposed method for different values of α even when $n = 8$.

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